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A MODIFIED KIRCHHOFF THEORY FOR BOUNDARY ELEMENT BENDING ANALYSIS OF THIN PLATES

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Abstract—This paper introduces a modified Kirchhoff theory in which the transverse normal stress is considered within the analysis of thin plates in bending. A consistent boundary element approach based upon three degrees-of-freedom, which avoids the development of Kirchhoff forces at plate corners, is presented for plates with arbitrary shapes and boundary conditions. Several case studies have been analysed and the results were compared with corresponding analytical solutions. It is clear that such an approach is accurate and easy to program. The transverse stress has little effect on the plate deflection, but it can be considered in strength assessment.

INTRODUCTION

The boundary element method (BEM) has been developed immensely in the last two decades and it has been extended to a large number of engineering applications, including the analysis of thin and thick plates in bending [see El-Zafrany (1993)]. An integral equation formulation for plate-bending problems was introduced by Jaswon and Maiti (1968). A presentation of this work with further developments was given later by Jaswon and Symm (1977). An indirect boundary integral equation approach to the solution of Kirchhoff plate-bending problems was presented by Altiero and Sikarskie (1978) and a similar analysis was given by Tottenham (1979). Direct formulations for thin plates were presented by Bezine and Gamby (1978) and Stern (1979). Several attempts aimed at improving the interpolation functions within boundary elements appeared later in the literature [see Hartmann and Zotemantel (1986)].

Most of the derivations found so far in the literature are based upon two degrees-of-freedom per boundary node. Such an approach usually results in the presence of additional terms, known as Kirchhoff forces, at the boundary corners. These terms are functions of boundary unknowns and contain singular functions, which makes their numerical treatment and programming too difficult, as demonstrated by Debbih (1987).

Consider a plate in bending, with a thickness h and a midplane in the x–y plane. According to the basic assumptions of the Kirchhoff theory, the lateral deflection w is considered independent of z, and the transverse stresses are ignored. For homogeneous, isotropic, elastic plate it can, therefore, be concluded that

$$\sigma_z = 0, \quad \varepsilon_z = 0$$

i.e.

$$\sigma_x + \sigma_y = \varepsilon_x + \varepsilon_y = 0,$$

or in other words

$$\nabla^2 w(x,y) = 0,$$

which is not an accurate governing equation for thin plates, as pointed out by McMillan (1988).

If σ_z is not zero, then the plane-stress equations adopted for thin plate analysis cannot be used, and if ε_z is not zero, then one should consider the lateral deflection being a function of (x, y, z).

This paper introduces a modified Kirchhoff theory in which the effect of the transverse stress σ_z is considered. A corresponding boundary element formulation based upon three degrees-of-freedom per node is presented. Some case studies are analysed so as to illustrate the merits of the theory presented.

REVIEW OF GOVERNING EQUATIONS

The analysis of thin plates in bending is based upon the assumptions of Kirchhoff's theory [see Debbih (1989)], with the exception that the effect of the transverse normal stress σ_z is considered. Considering a plate, as described earlier, with the surface $z = -h/2$ subjected to a lateral force of intensity p, a distribution for σ_z over the thickness can be written as follows [see Reissner (1945)]:

$$\sigma_z = \frac{p}{2}\left(-1+\frac{3z}{h}-\frac{4z^3}{h^3}\right), \tag{1}$$

where h is the plate thickness. The bending strain components can be approximated at any point (x, y, z) inside the plate as follows:

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}, \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2}, \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{2}$$

and the other components of strain are negligible.

The generalized Hooke's law can be written, for such a case, in the following form:

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) - \frac{\nu}{E}\sigma_z \\ \varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) - \frac{\nu}{E}\sigma_z \\ \gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy}, \end{aligned} \tag{3}$$

where E is Young's modulus of elasticity and ν is Poisson's ratio. Hence, it can be deduced from eqns (2) and (3) that

$$\begin{aligned} \sigma_x &= -\frac{Ez}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+\frac{\nu}{(1-\nu)}\sigma_z \\ \sigma_y &= -\frac{Ez}{(1-\nu^2)}\left(\nu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\frac{\nu}{(1-\nu)}\sigma_z \\ \tau_{xy} &= -\frac{Ez}{(1+\nu)}\frac{\partial^2 w}{\partial x \partial y}. \end{aligned} \tag{4}$$

Bending moments per unit length are usually defined as follows:

$$\{M_x M_y M_{xy}\} = \int_{-h/2}^{h/2} z\{\sigma_x \sigma_y \tau_{xy}\}\,\mathrm{d}z \tag{5}$$

and it can, therefore, be shown that

$$\begin{aligned} M_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) + \frac{vp}{(1-v)\lambda^2} \\ M_y &= -D\left(v\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{vp}{(1-v)\lambda^2} \\ M_{xy} &= -(1-v)D\frac{\partial^2 w}{\partial x \partial y}, \end{aligned} \tag{6}$$

where

$$D = \frac{Eh^3}{12(1-v^2)}, \lambda^2 = \frac{10}{h^2}.$$

Equations of equilibrium over the plate thickness can be written in the following form:

$$\begin{aligned} \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= 0 \\ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y &= 0 \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p &= 0, \end{aligned} \tag{7}$$

where Q_x and Q_y represent shear forces per unit length, and it can deduced from eqns (6) and (7) that

$$Q_x = -D\frac{\partial}{\partial x}(\nabla^2 w) + \frac{v}{(1-v)\lambda^2}\frac{\partial p}{\partial x} \tag{8a}$$

$$Q_y = -D\frac{\partial}{\partial y}(\nabla^2 w) + \frac{v}{(1-v)\lambda^2}\frac{\partial p}{\partial y} \tag{8b}$$

and

$$D\nabla^4 w = p + \frac{v}{(1-v)\lambda^2}\nabla^2 p. \tag{9}$$

DERIVATION OF BOUNDARY INTEGRAL EQUATIONS

Consider a plate with its midplane defined in the x–y plane by means of a two-dimensional domain Ω, which has a boundary Γ. If there exists a solution $w(x, y)$ which satisfies the given boundary conditions, a weighted-residual expression can be deduced from eqns (7) as follows:

$$\iint_\Omega \left[\theta_x^*\left(\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x\right)+\theta_y^*\left(\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y\right)\right.$$
$$\left.+w^*\left(\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+p\right)\right]\mathrm{d}x\,\mathrm{d}y=0, \qquad (10)$$

where $\theta_x^*, \theta_y^*, w^*$ are weighting functions, and since we have one unknown function $w(x,y)$, then those weighting functions should be related.

Using integration-by-parts theorems listed by El-Zafrany (1993), the previous expression can be reduced to the following weak form:

$$\oint_\Gamma \left[\theta_x^*(lM_x+mM_{xy})+\theta_y^*(lM_{xy}+mM_y)+w^*\,(lQ_x+mQ_y)\right]\mathrm{d}\Gamma$$
$$+\iint_\Omega w^*p\,\mathrm{d}x\,\mathrm{d}y-\iint_\Omega\left[Q_x\left(\theta_x^*+\frac{\partial w^*}{\partial x}\right)+Q_y\left(\theta_y^*+\frac{\partial w^*}{\partial y}\right)\right]\mathrm{d}x\,\mathrm{d}y$$
$$-\iint_\Omega\left[M_x\frac{\partial\theta_x^*}{\partial x}+M_{xy}\left(\frac{\partial\theta_x^*}{\partial y}+\frac{\partial\theta_y^*}{\partial x}\right)+M_y\frac{\partial\theta_y^*}{\partial y}\right]\mathrm{d}x\,\mathrm{d}y=0, \qquad (11)$$

where (l,m) are the directional cosines of the outward normal to the boundary Γ. Defining the functions M_x^*, M_y^*, M_{xy}^* such that

$$M_x^*=D\left(\frac{\partial\theta_x^*}{\partial x}+\nu\frac{\partial\theta_y^*}{\partial y}\right)$$
$$M_y^*=D\left(\nu\frac{\partial\theta_x^*}{\partial x}+\frac{\partial\theta_y^*}{\partial y}\right)$$
$$M_{xy}^*=\frac{(1-\nu)}{2}D\left(\frac{\partial\theta_x^*}{\partial y}+\frac{\partial\theta_y^*}{\partial x}\right), \qquad (12)$$

then it can be proved from eqns (6) and (12) that

$$M_x\frac{\partial\theta_x^*}{\partial x}+M_{xy}\left(\frac{\partial\theta_x^*}{\partial y}+\frac{\partial\theta_y^*}{\partial x}\right)+M_y\frac{\partial\theta_y^*}{\partial y}$$
$$\equiv M_x^*\frac{\partial\theta_x}{\partial x}+M_{xy}^*\left(\frac{\partial\theta_x}{\partial y}+\frac{\partial\theta_y}{\partial x}\right)+M_y^*\frac{\partial\theta_y}{\partial y}+\frac{\nu p}{(1-\nu)\lambda^2}\left[\frac{\partial\theta_x^*}{\partial x}+\frac{\partial\theta_y^*}{\partial y}\right],$$

where

$$\theta_x=-\frac{\partial w}{\partial x},\quad \theta_y=-\frac{\partial w}{\partial y}.$$

Hence, eqn (11) can be modified as follows:

$$\oint_\Gamma(\theta_x^*t_x+\theta_y^*t_y+w^*t_z)\,\mathrm{d}\Gamma+\iint_\Omega p\left[w^*-\frac{\nu}{(1-\nu)\lambda^2}\left(\frac{\partial\theta_x^*}{\partial x}+\frac{\partial\theta_y^*}{\partial y}\right)\right]\mathrm{d}x\,\mathrm{d}y$$
$$-\iint_\Omega\left[\left(\frac{\partial\theta_x}{\partial x}M_x^*+\frac{\partial\theta_x}{\partial y}M_{xy}^*\right)+\left(\frac{\partial\theta_y}{\partial x}M_{xy}^*+\frac{\partial\theta_y}{\partial y}M_y^*\right)\right]\mathrm{d}x\,\mathrm{d}y$$
$$-\iint_\Omega\left[Q_x\left(\theta_x^*+\frac{\partial w^*}{\partial x}\right)+Q_y\left(\theta_y^*+\frac{\partial w^*}{\partial y}\right)\right]\mathrm{d}x\,\mathrm{d}y=0, \qquad (13)$$

where

$$t_x = lM_x + mM_{xy}$$
$$t_y = lM_{xy} + mM_y$$
$$t_z = lQ_x + mQ_y. \tag{14}$$

Equation (13) can be simplified by using the following definitions for θ_x^*, θ_y^*:

$$\theta_x^* = -\frac{\partial w^*}{\partial x}, \quad \theta_y^* = -\frac{\partial w^*}{\partial y}, \tag{15}$$

and by integrating the resulting expression by parts once more, then eqn (13) can be reduced to

$$\oint_\Gamma (t_x\theta_x^* + t_y\theta_y^* + t_z w^*)\,\mathrm{d}\Gamma - \oint_\Gamma (t_x^*\theta_x + t_y^*\theta_y)\,\mathrm{d}\Gamma + \iint_\Omega p\left[w^* + \frac{v}{(1-v)\lambda^2}\nabla^2 w^*\right]\mathrm{d}x\,\mathrm{d}y$$
$$+\iint_\Omega \left[\theta_x\left(\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y}\right) + \theta_y\left(\frac{\partial M_{xy}^*}{\partial x} + \frac{\partial M_y^*}{\partial y}\right)\right]\mathrm{d}x\mathrm{d}y = 0, \tag{16}$$

where

$$t_x^* = lM_x^* + mM_{xy}^*, \quad t_y^* = lM_{xy}^* + mM_y^*. \tag{17}$$

Defining two new functions ϕ_x^*, ϕ_y^* such that

$$\phi_x^* = \frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} \equiv -D\frac{\partial}{\partial x}(\nabla^2 w^*)$$
$$\phi_y^* = \frac{\partial M_{xy}^*}{\partial x} + \frac{\partial M_y^*}{\partial y} \equiv -D\frac{\partial}{\partial y}(\nabla^2 w^*), \tag{18}$$

and noticing that

$$\iint_\Omega (\phi_x^*\theta_x + \phi_y^*\theta_y)\,\mathrm{d}x\mathrm{d}y \equiv -\oint_\Gamma t_z^* w\,\mathrm{d}\Gamma + \iint_\Omega w\left(\frac{\partial \phi_x^*}{\partial x} + \frac{\partial \phi_y^*}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y, \tag{19}$$

where $t_z^* = l\phi_x^* + m\phi_y^*$, then eqn (16) can be reduced to the following form:

$$\oint_\Gamma (t_x\theta_x^* + t_y\theta_y^* + t_z w^*)\,\mathrm{d}\Gamma - \oint_\Gamma (t_x^*\theta_x + t_y^*\theta_y + t_z^* w)\,\mathrm{d}\Gamma$$
$$+\iint_\Omega p\left[w^* + \frac{v}{(1-v)\lambda^2}\nabla^2 w^*\right]\mathrm{d}x\,\mathrm{d}y + \iint_\Omega w\left(\frac{\partial \phi_x^*}{\partial x} + \frac{\partial \phi_y^*}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = 0. \tag{20}$$

Let the weighting function w^* be defined, with respect to a point (x_i, y_i), such that

$$\frac{\partial \phi_x^*}{\partial x} + \frac{\partial \phi_y^*}{\partial y} = -\left(e_x\frac{\partial}{\partial x} + e_y\frac{\partial}{\partial y} + e_z\right)\delta(x-x_i, y-y_i), \tag{21}$$

where e_x, e_y, e_z are arbitrary parameters, the point (x_i, y_i) is known as the *source point*, and $\delta(x-x_i, y-y_i)$ is a two-dimensional Dirac delta function. Hence, by using the properties of the delta functions (El-Zafrany, 1993), eqn (20) can be reduced to the following boundary integral equation:

$$c_i[e_x\theta_x(x_i,y_i)+e_y\theta_y(x_i,y_i)+e_z w(x_i,y_i)]+\oint_\Gamma (t_x^*\theta_x+t_y^*\theta_y+t_z^* w)\,\mathrm{d}\Gamma$$

$$=\oint_\Gamma (t_x\theta_x^*+t_y\theta_y^*+t_z w^*)\,\mathrm{d}\Gamma+\iint_\Omega p\left[w^*+\frac{\nu}{(1-\nu)\lambda^2}\nabla^2 w^*\right]\mathrm{d}x\,\mathrm{d}y \quad (22)$$

where

$$c_i \equiv \iint_\Omega \delta(x-x_i,y-y_i)\mathrm{d}x\,\mathrm{d}y.$$

To facilitate the application of boundary conditions for plates with arbitrary shapes, eqn (22) may also be rewritten in the following form:

$$c_i[e_x\theta_x(x_i,y_i)+e_y\theta_y(x_i,y_i)+e_z w(x_i,y_i)]+\oint_\Gamma (M_n^*\theta_n+M_{nt}^*\theta_t+Q_n^* w)\,\mathrm{d}\Gamma$$

$$=\oint_\Gamma (M_n\theta_n^*+M_{nt}\theta_t^*+Q_n w^*)\,\mathrm{d}\Gamma+\iint_\Omega p\left[w^*+\frac{\nu}{(1-\nu)\lambda^2}\nabla^2 w^*\right]\mathrm{d}x\,\mathrm{d}y, \quad (23)$$

where

$$M_n = lt_x+mt_y, \quad M_{nt} = -mt_x+lt_y, \quad Q_n \equiv t_z$$

and similar definitions are employed for the starred parameters.

FUNDAMENTAL SOLUTION PARAMETERS

Defining a function f^* such that

$$w^* = \left(e_x\frac{\partial}{\partial x}+e_y\frac{\partial}{\partial y}+e_z\right)f^*, \quad (24)$$

then it can be deduced from eqns (18), (21) and (24) that

$$D\left(e_x\frac{\partial}{\partial x}+e_y\frac{\partial}{\partial y}+e_z\right)\nabla^4 f^* = \left(e_x\frac{\partial}{\partial x}+e_y\frac{\partial}{\partial y}+e_z\right)\delta(x-x_i,y-y_i), \quad (25)$$

which is satisfied for arbitrary values of e_x, e_y and e_z, if f^* satisfies the following equation:

$$D\nabla^4 f^* = \delta(x-x_i,y-y_i). \quad (26)$$

The previous equation has the following general solution [see El-Zafrany (1993)]:

$$f^* = \frac{r^2}{8\pi D}(\log r-1+c_1)+c_2, \quad (27)$$

where c_1, c_2 are arbitrary integration constants and

$$r = \sqrt{(x-x_i)^2+(y-y_i)^2}.$$

It can be deduced from eqn (24) and the definitions of other parameters, that

$$\begin{bmatrix} \theta_n^* \\ \theta_t^* \\ w^* \end{bmatrix} = \mathbf{U} \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \tag{28}$$

and

$$\begin{bmatrix} M_n^* \\ M_{nt}^* \\ Q_n^* \end{bmatrix} = \mathbf{T} \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}, \tag{29}$$

where **U** and **T** are 3×3 matrices defined as follows:

$$\mathbf{U} = \begin{bmatrix} -\dfrac{\partial^2}{\partial x\,\partial n} & -\dfrac{\partial^2}{\partial y\,\partial n} & -\dfrac{\partial}{\partial n} \\ -\dfrac{\partial^2}{\partial x\,\partial t} & -\dfrac{\partial^2}{\partial y\,\partial t} & -\dfrac{\partial}{\partial t} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 1 \end{bmatrix} f^* \tag{30}$$

$$\mathbf{T} = -D \begin{bmatrix} \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2}{\partial n^2}+v\dfrac{\partial^2}{\partial t^2}\right) & \dfrac{\partial}{\partial y}\left(\dfrac{\partial^2}{\partial n^2}+v\dfrac{\partial^2}{\partial t^2}\right) & \left(\dfrac{\partial^2}{\partial n^2}+v\dfrac{\partial^2}{\partial t^2}\right) \\ (1-v)\dfrac{\partial^3}{\partial x\,\partial n\,\partial t} & (1-v)\dfrac{\partial^3}{\partial y\,\partial n\,\partial t} & (1-v)\dfrac{\partial^2}{\partial n\,\partial t} \\ \dfrac{\partial^2}{\partial x\,\partial n}\nabla^2 & \dfrac{\partial^2}{\partial y\,\partial n}\nabla^2 & \dfrac{\partial}{\partial n}\nabla^2 \end{bmatrix} f^*. \tag{31}$$

Notice also that

$$\frac{\partial f}{\partial n} \equiv \frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} \tag{32}$$

$$\frac{\partial f}{\partial t} \equiv -m\frac{\partial f}{\partial x} + l\frac{\partial f}{\partial y} \tag{33}$$

and, if $\hat{n}_0$ is a unit vector defined in terms of its directional cosines as follows,

$$\hat{n}_0 = l_0\hat{i} + m_0\hat{j},$$

then the rate of change of a function f with respect to a length measured in $\hat{n}_0$ direction can be expressed as

$$\frac{\partial f}{\partial n_0} = \nabla f \cdot \hat{n}_0 \equiv l_0\frac{\partial f}{\partial x} + m_0\frac{\partial f}{\partial y}. \tag{34}$$

The following theorems were employed for the derivation of explicit forms for **U** and **T** matrices:

$$\frac{\partial f^*}{\partial n_0} = \frac{r}{4\pi D}(\log r - \tfrac{1}{2} + c_1)\frac{\partial r}{\partial n_0} \tag{35}$$

$$\frac{\partial^2 f^*}{\partial n_1 \partial n_2} = \frac{1}{4\pi D}\left[(\hat{n}_1 \cdot \hat{n}_2)(\log r - \tfrac{1}{2} + c_1) + \frac{\partial r}{\partial n_1}\frac{\partial r}{\partial n_2}\right] \tag{36}$$

$$\frac{\partial^3 f^*}{\partial n_1 \partial n_2 \partial n_3} = \frac{1}{4\pi Dr}\left[(\hat{n}_2 \cdot \hat{n}_3)\frac{\partial r}{\partial n_1} + (\hat{n}_3 \cdot \hat{n}_1)\frac{\partial r}{\partial n_2} + (\hat{n}_1 \cdot \hat{n}_2)\frac{\partial r}{\partial n_3} - 2\frac{\partial r}{\partial n_1}\frac{\partial r}{\partial n_2}\frac{\partial r}{\partial n_3}\right], \tag{37}$$

and a listing of those explicit expressions, based upon $c_1 = c_2 = 0$, is in the Appendix.

REDUCTION OF BOUNDARY INTEGRAL EQUATIONS

If a function h^* is defined such that

$$\nabla^2 h^* = f^* \equiv \frac{1}{r}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}h^*}{\mathrm{d}r}\right)\right],$$

then it can be deduced by direct integration, and ignoring the resulting integration constants, that

$$h^* = \frac{r^4}{128\pi D}(\log r - \tfrac{3}{2} + c_1) + \frac{r^2}{4}c_2. \tag{38}$$

For arbitrary values of e_x, e_y and e_z, and cases with uniformly- or linearly-distributed loading, one can reduce eqn (23) to the following simultaneous boundary integral equations, with respect to the source point (x_i, y_i):

$$c_{11}\theta_x(x_i, y_i) + c_{12}\theta_y(x_i, y_i) + c_{13}w(x_i, y_i) + \oint_\Gamma (T_{11}\theta_n + T_{21}\theta_t + T_{31}w)\mathrm{d}\Gamma$$
$$= \oint_\Gamma (U_{11}M_n + U_{21}M_{nt} + U_{31}Q_n)\mathrm{d}\Gamma + \oint_\Gamma \left(C_x^* p - L_x^* \frac{\partial p}{\partial n}\right)\mathrm{d}\Gamma \tag{39a}$$

$$c_{21}\theta_x(x_i, y_i) + c_{22}\theta_y(x_i, y_i) + c_{23}w(x_i, y_i) + \oint_\Gamma (T_{12}\theta_n + T_{22}\theta_t + T_{32}w)\mathrm{d}\Gamma$$
$$= \oint_\Gamma (U_{12}M_n + U_{22}M_{nt} + U_{32}Q_n)\mathrm{d}\Gamma + \oint_\Gamma \left(C_y^* p - L_y^* \frac{\partial p}{\partial n}\right)\mathrm{d}\Gamma \tag{39b}$$

$$c_{31}\theta_x(x_i, y_i) + c_{32}\theta_y(x_i, y_i) + c_{33}w(x_i, y_i) + \oint_\Gamma (T_{13}\theta_n + T_{23}\theta_t + T_{33}w)\mathrm{d}\Gamma$$
$$= \oint_\Gamma (U_{13}M_n + U_{23}M_{nt} + U_{33}Q_n)\mathrm{d}\Gamma + \oint_\Gamma \left(C_z^* p - L_z^* \frac{\partial p}{\partial n}\right)\mathrm{d}\Gamma, \tag{39c}$$

where

$$C_x^* = \frac{\partial^2 g^*}{\partial x \, \partial n}, \quad C_y^* = \frac{\partial^2 g^*}{\partial y \, \partial n}, \quad C_z^* = \frac{\partial g^*}{\partial n} \tag{40}$$

$$L_x^* = \frac{\partial g^*}{\partial x}, \quad L_y^* = \frac{\partial g^*}{\partial y}, \quad L_z^* = g^* \tag{41}$$

and

$$g^* = h^* + \frac{\nu}{(1-\nu)\lambda^2} f^*. \tag{42}$$

Notice also that additional terms are added to take into consideration corner effects or jump functions [see Husain (1989)], and $\tilde{\Gamma}$ represents the boundary without corners, i.e. without any corner effect on contour integrals.

If there exists a concentrated shear force F_z acting at a point (x_j, y_j) inside the domain Ω, then a two-dimensional Dirac delta function can be employed to define an equivalent distributed loading q, as follows:

$$q = F_z \delta(x - x_j, y - y_j). \tag{43}$$

Hence, the corresponding domain integrals can be eliminated and represented by simple terms, using the following property of the Dirac delta function [see El-Zafrany (1993)]:

$$\iint_\Omega f(x,y)\delta(x-x_j, y-y_j) \, \mathrm{d}x \, \mathrm{d}y = f(x_j, y_j). \tag{44}$$

Internal stresses and moments can be calculated from boundary integral equations derived from eqns (39), and further details about the numerical treatment of the boundary integral equations for such problems can be found in Debbih (1989).

APPLICATIONS

A FORTRAN program was developed using the modified Kirchhoff theory and constant boundary elements. Several examples demonstrating different boundary and loading conditions were analysed using the developed programs on a PC, and the boundary element results were compared with corresponding analytical solutions, as summarized next.

Clamped circular plate under uniformly-distributed loading

A circular plate with the following properties was considered:

outer radius	$r_0 = 0.5\,\mathrm{m}$
plate thickness	$h = 0.05\,\mathrm{m}$
Young's modulus	$E = 2.1 \times 10^8\,\mathrm{kN/m^2}$
Poisson's ratio	$\nu = 0.3$
loading intensity	$p = 1200\,\mathrm{kN/m^2}$.

The results are represented in terms of non-dimensional parameters, where

$$w_0 = \frac{qr_0^4}{D}, \quad M_0 = qr_0^2.$$

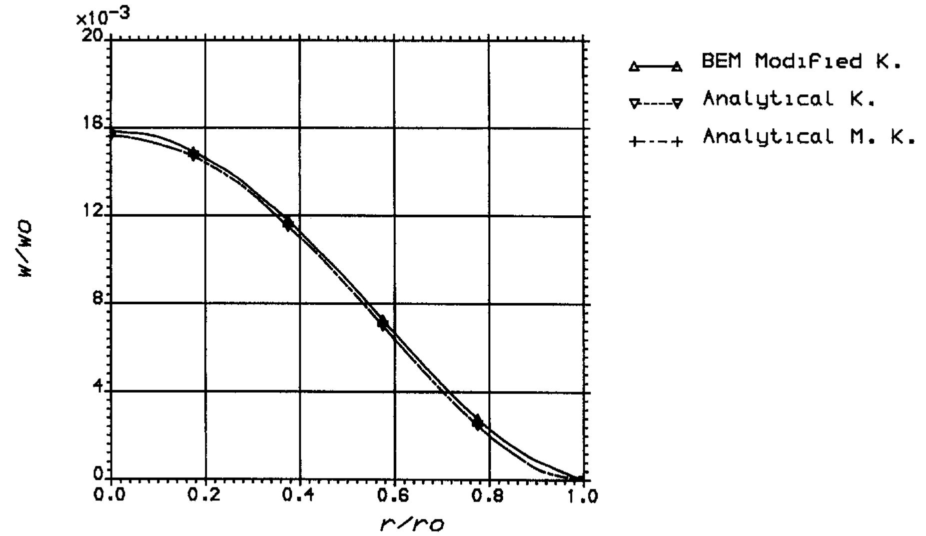

Fig. 1. Deflection of clamped circular plate under uniformly-distributed loading.

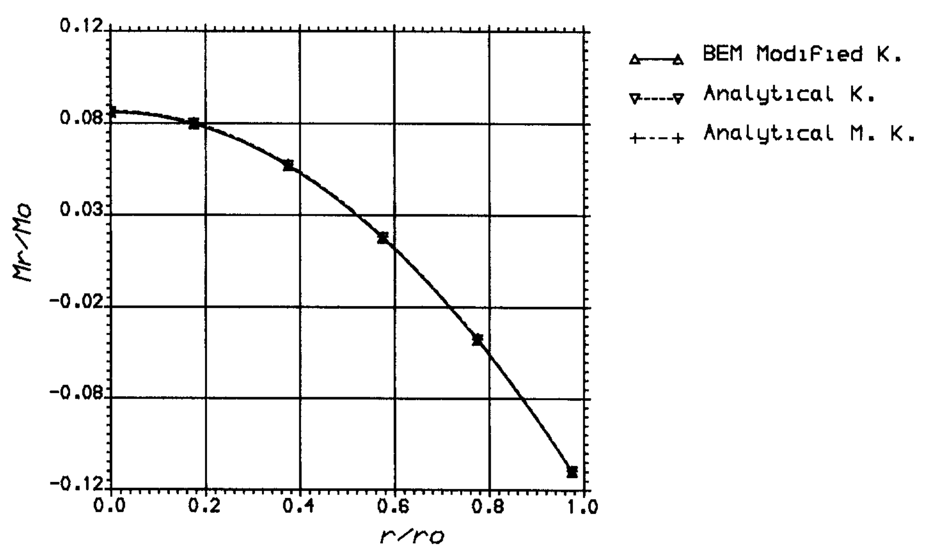

Fig. 2. Moment M_r for a clamped circular plate under uniformly-distributed loading.

Boundary element results were evaluated at internal nodes on a radial line from the centre of the plate to its outer surface, and the resulting parameters were compared with corresponding analytical solutions given by Debbih (1989), as shown in Figs 1 and 2, which illustrate non-dimensional radial distributions of the lateral deflection w and the moment M_r, respectively. The values of w/w_0 and M_r/M_0 at the plate centre, as obtained from boundary element results and analytical solutions, are also demonstrated in Table 1. It is

Table 1. Non-dimensional deflection and moment at the centre of the clamped circular plate

	Analytical Kirchhoff	Analytical Mod. Kirchhoff	BEM Kirchhoff	BEM Mod. Kirchhoff
w/w_0	0.0156	0.0156	0.0165	0.0159
M_r/M_0	0.8125	0.8168	0.8245	0.8087

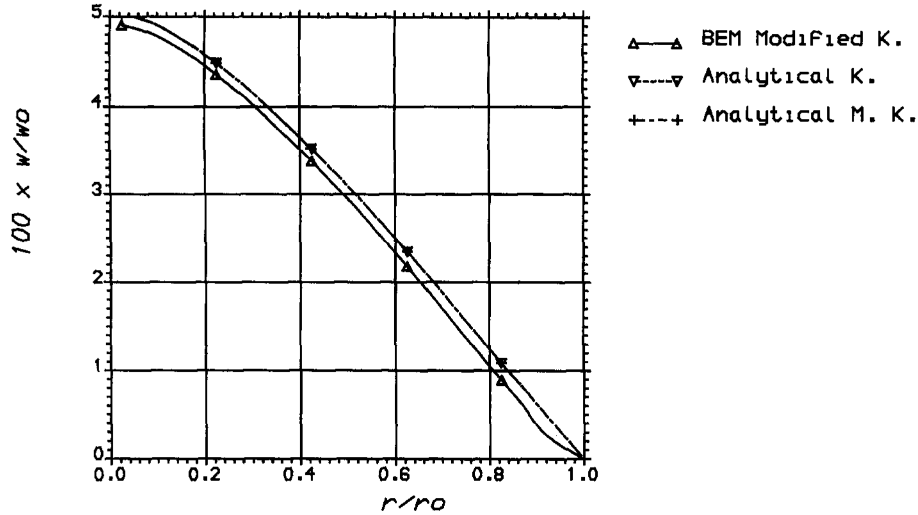

Fig. 3. Deflection of simply-supported circular plate under concentrated loading.

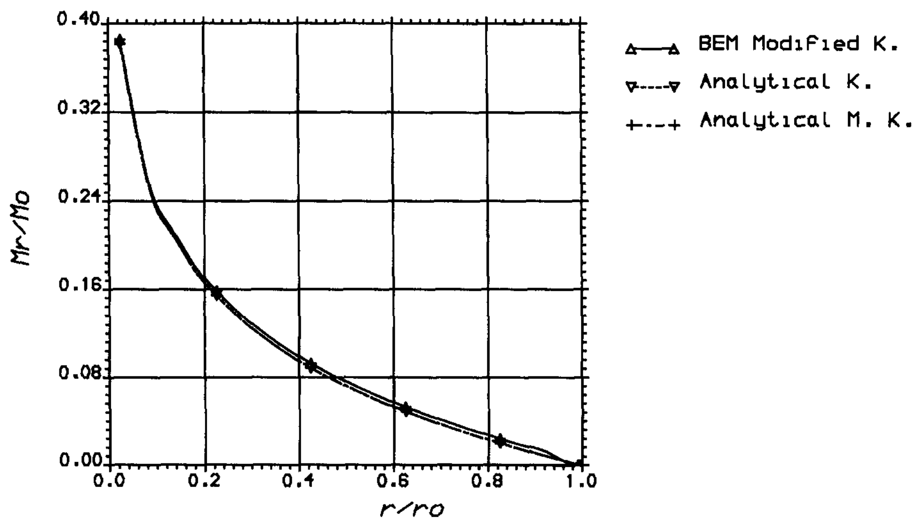

Fig. 4. Moment M_r of simply-supported circular plate under concentrated loading.

clear that the BEM results agree very well with analytical solutions, and the modified theory has led to accurate boundary element results.

Simply-supported circular plate under concentrated loading

This case is similar to the previous one but the plate is subjected to a concentrated shear force at its centre, $F = 300$ kN, and

$$w_0 = \frac{Fr_0^2}{D}, \quad M_0 \equiv F.$$

Non-dimensional radial distributions of w and M_r are shown in Figs 3 and 4, respectively. This case proves that the procedure suggested for dealing with concentrated forces is accurate, and one can obtain a value of M_r approaching infinity at the point of load application.

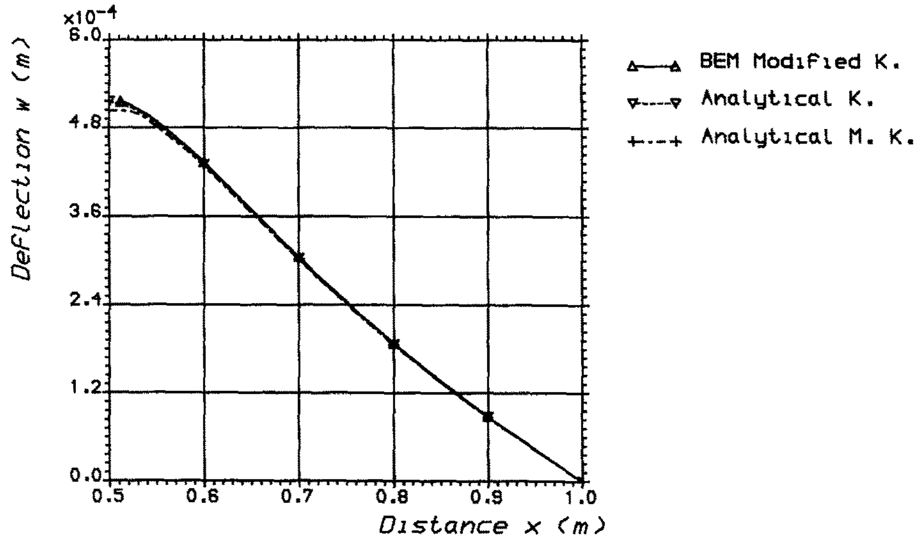

Fig. 5. Deflection of simply-supported rectangular plate under concentrated loading.

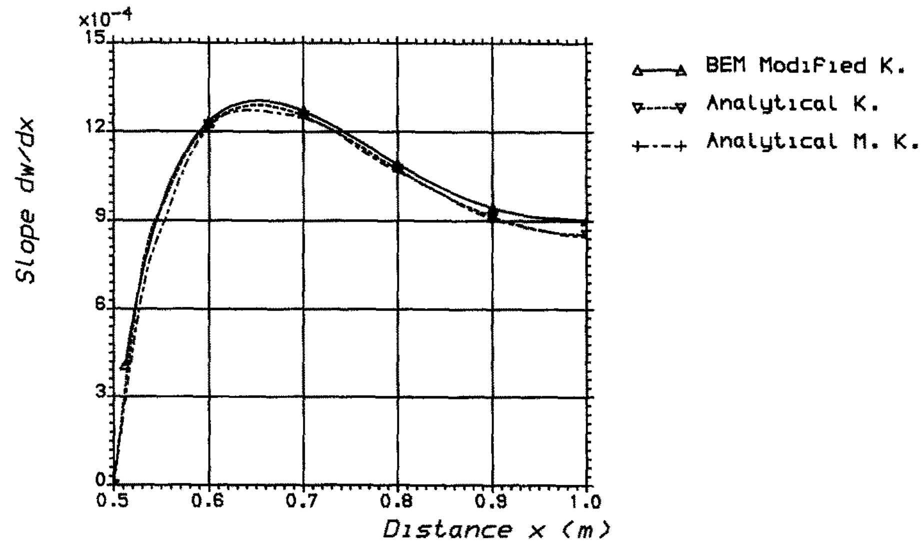

Fig. 6. Slope of simply-supported rectangular plate under concentrated loading.

Simply-supported rectangular plate under concentrated loading

This case represents a rectangular plate with its midplane defined by means of the following domain:

$$\Omega = \{(x, y); \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 0.5\}.$$

The four sides of the plate are simply supported, i.e. $M_n = M_{nt} = 0$, $w = 0$ over the boundary of the plate, and other properties are similar to the previous case.

Boundary element results are considered at internal nodes on the line $y = 0.25$, and they are plotted against corresponding analytical solutions, as shown in Figs 5, 6 and 7, which indicate the distribution of the deflection w, slope $\partial w/\partial x$ and moment M_x, respectively. The displayed boundary element results prove to be very accurate compared with analytical solutions, and hence this case demonstrates the accuracy and efficiency of the BEM based upon three degrees-of-freedom for plates with corners.

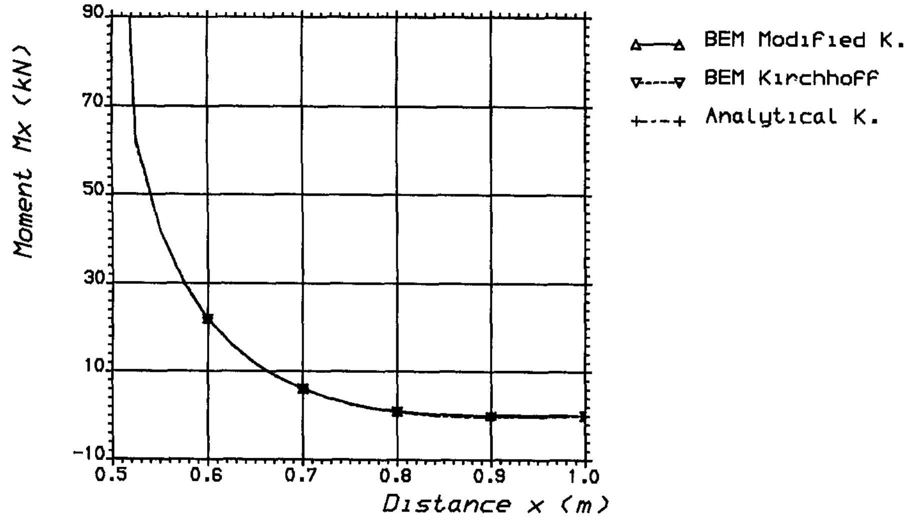

Fig. 7. Moment M_x of simply-supported rectangular plate under concentrated loading.

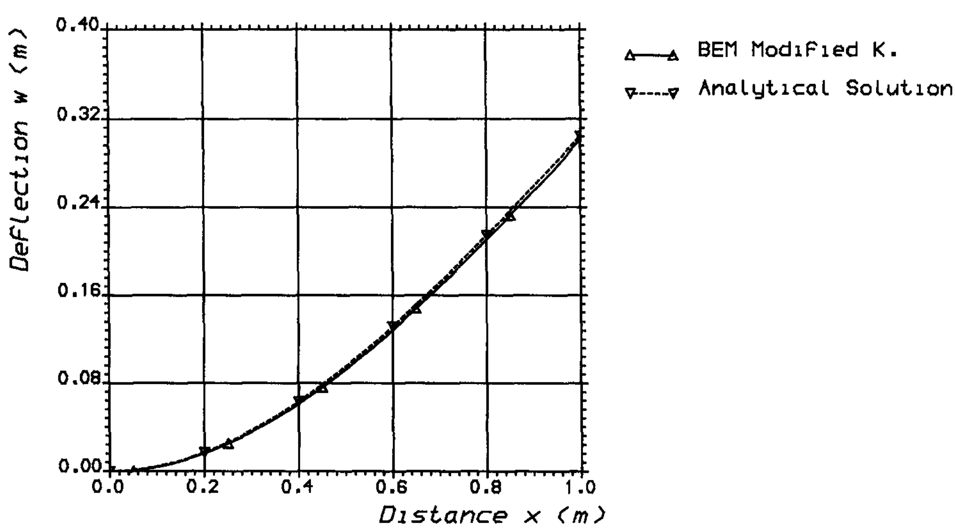

Fig. 8. Deflection of rectangular plate under corner forces.

Rectangular plate under corner forces

This case is similar in geometry and material properties to the previous case, but it has one of the shortest edges ($x = 0$) clamped, and all other edges are free. The free corners are subjected to two concentrated forces, each of value $F = 500$ kN. The significance of such a case for the BEM is that it has corners, free edges and singular corner forces, which allow the terms of the boundary integrals containing singular and divergent integrals to be tested. The deflection distribution along the centre line ($y = 0.25$) is plotted against an analytical solution based upon engineering beam theory, as shown in Fig. 8, which proves the accuracy and reliability of the BEM formulations presented in this work.

Simply-supported rhombic plate under uniformly-distributed loading

This case has been selected to test the performance of the presented boundary element theory in the analysis of skew plates. It represents a simply-supported rhombic plate with thickness $h = 0.05$ m, and a midplane as shown in Fig. 9. The Young's modulus of the plate material is $3.0 \times 10^7 \text{kN/m}^2$, its Poisson's ratio is equal to 0.2, and the plate is subjected to a uniformly-distributed loading of intensity $p = 5000 \text{ kN/m}^2$.

The maximum deflection and moment, which occur at the plate centre, are represented as follows [see Timoshenko and Krieger (1970)]:

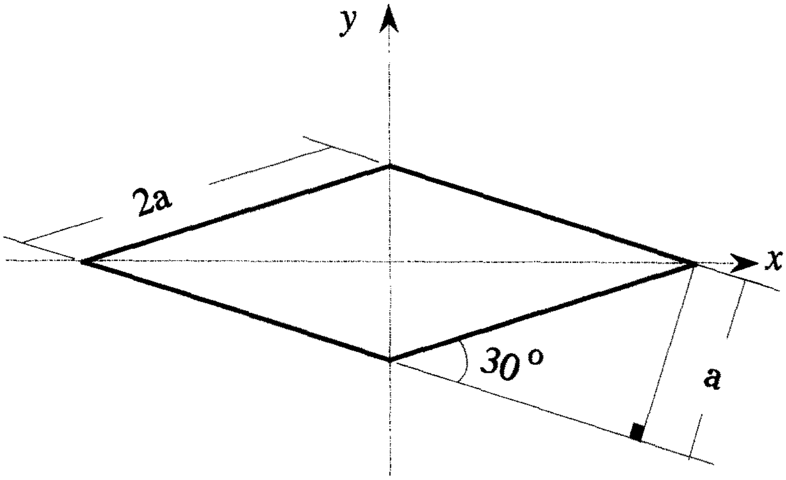

Fig. 9. Midplane of rhombic plate.

Table 2. Coefficients for maximum deflection and bending moment for the rhombic plate

	α	β
Timoshenko and Krieger (1970)	0.00796	0.0772
Boundary element analysis	0.00788	0.0775

$$w_{\max} = \alpha \frac{qa^4}{D}, \quad M_{\max} = \beta q a^2.$$

The values of α, β as obtained from boundary element analysis are tabulated against corresponding published values, as shown in Table 2, which proves that the boundary element theory presented in this work can be employed accurately for the analysis of skew plates.

CONCLUSIONS

It is clear from the presented results that the BEM derivations based upon three degrees-of-freedom are accurate and have led to simple programs avoiding the estimation of unknown Kirchhoff forces at the plate corners, and yielding very accurate results for plates with free-edge conditions irrespective of the divergent integrals encountered in the analysis. The effect of the transverse stress on deflection and moments is very small, but the stress is evaluated during the BEM analysis and can, therefore, be considered in the assessment of the plate strength. The methodology employed in the derivation of fundamental solutions and boundary integral equations is consistent and much simpler than any existing publication in this area. Using such a simplified theory, the authors have managed to develop simple boundary element programs which run successfully on PCs.

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APPENDIX

*Terms of the **U** matrix*

$$U_{\alpha\beta} = -\frac{1}{8\pi D}\left[(\hat{n}_\alpha\cdot\hat{i}_\beta)(2\log r-1)+2\frac{\partial r}{\partial n_\alpha}\frac{\partial r}{\partial x_\beta}\right]$$

$$U_{\alpha 3} = -\frac{r}{8\pi D}\frac{\partial r}{\partial n_\alpha}(2\log r-1)$$

$$U_{3\beta} = \frac{r}{8\pi D}\frac{\partial r}{\partial x_\beta}(2\log r-1)$$

$$U_{33} = \frac{r^2}{8\pi D}(\log r-1).$$

*Terms of the **T** matrix*

$$T_{1\beta} = -\frac{1}{4\pi r}\left[(1+\nu)\frac{\partial r}{\partial x_\beta}+2(1-\nu)\frac{\partial r}{\partial n}\left(\hat{n}\cdot\hat{i}_\beta-2\frac{\partial r}{\partial n}\frac{\partial r}{\partial x_\beta}\right)\right]$$

$$T_{2\beta} = -\frac{(1-\nu)}{4\pi r}\left[(\hat{t}\cdot\hat{i}_\beta)\frac{\partial r}{\partial n}+(\hat{n}\cdot\hat{i}_\beta)\frac{\partial r}{\partial t}-2\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}\frac{\partial r}{\partial x_\beta}\right]$$

$$T_{3\beta} = -\frac{1}{4\pi r^2}\left[(\hat{n}\cdot\hat{i}_\beta)-2\frac{\partial r}{\partial n}\frac{\partial r}{\partial x_\beta}\right]$$

$$T_{13} = -\frac{1}{4\pi}\left[(1+\nu)(\log r-\tfrac{1}{2})+\nu+(1-\nu)\left(\frac{\partial r}{\partial n}\right)^2\right]$$

$$T_{23} = -\frac{(1-\nu)}{4\pi}\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}$$

$$T_{33} = -\frac{1}{2\pi r}\frac{\partial r}{\partial n},$$

where

$$\alpha = 1,2,\quad \beta = 1,2$$

$$(x_1,x_2)\equiv(x,y),\quad (\hat{i}_1,\hat{i}_2)\equiv(\hat{i},\hat{j})$$

$$\hat{n}_1\equiv\hat{n} = l\hat{i}+m\hat{j},\quad \hat{n}_2\equiv\hat{t} = -m\hat{i}+l\hat{j}$$

$$\frac{\partial r}{\partial n_1}\equiv\frac{\partial r}{\partial n} = (\nabla r)\cdot\hat{n},\quad \frac{\partial r}{\partial n_2}\equiv\frac{\partial r}{\partial t} = (\nabla r)\cdot\hat{t}.$$